Question Sheet 10, Riemann Integration. v1 2020-21

1. Let $f(x) = x^3$ on $[0, 1]$ and let $\mathcal{P}_n$ be the arithmetic partition that splits $[0, 1]$ into n equal subintervals.

   Evaluate $U(\mathcal{P}_n, f)$ and $L(\mathcal{P}_n, f)$.

   Thus show that f is Riemann integrable on $[0, 1]$ and find the value of

$$\int_0^1 x^3 dx.$$

   You may need to recall $\sum_{i=1}^{n} i^3 = n^2 (n+1)^2 /4$.

2. i) Integrate $f(x) = x^2$ over $[1, 2]$ by using the arithmetic partition of $[1, 2]$ into n equal subintervals.

   ii) Integrate $f(x) = x^2$ over $[1, 2]$ by using the geometric partition

$$\mathcal{Q}_n = \left\{1, \eta, \eta^2, \eta^3, ..., \eta^n = 2\right\},$$

   where η is the n^{th}-root of 2.

3. Integrate $f(x) = 1/x^3$ over $[2, 3]$ by using the geometric partition

$$\mathcal{Q}_n = \left\{2, 2\eta, 2\eta^2, 2\eta^3, ..., 2\eta^n = 3\right\},$$

   where η is the n^{th}-root of $3/2$.

4. i) If the function $h : [a, b] \to \mathbb{R}$ is bounded, Riemann integrable and satisfies $h(x) \geq 0$ for all $x \in [a, b]$, show that

$$\int_a^b h(x)\, dx \geq 0.$$

   Hint What does $h(x) \geq 0$ for all $x \in [a, b]$ say about any Lower Sum? What does it then say about the Lower Integral of h? Use also the fact that h is Riemann integrable implies that the lower and upper integrals both exist and are equal.

ii) Prove that if the functions f and g, are bounded on $[a,b]$, and satisfy $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g \quad \text{and} \quad \overline{\int_a^b} f \leq \overline{\int_a^b} g.$$

iii) Prove that if the Riemann integrable functions f and g satisfy $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f \leq \int_a^b g.$$

5. Integrate $f(x) = x^2 - x$ over $[2,5]$ by using

i) the arithmetic partition of $[2,5]$ into n equal length subintervals and

ii) the geometric partition of $[2,5]$ into n intervals.

6. **Definition** *If f is continuous on $(a,b)$and F is continuous on $[a,b]$ and and differentiable on (a,b) with $F'(x) = f(x)$ for all $x \in (a,b)$ then F is a* ***primitive*** *for f.*

Find primitives for

(i) $\dfrac{1}{\sqrt{1-x^2}}$, (ii) $\dfrac{x}{\sqrt{1-x^2}}$, (iii) $\dfrac{1}{\sqrt{1+x^2}}$.

iv) $\dfrac{x}{\sqrt{1+x^2}}$, v) $\dfrac{1}{1+x^2}$, vi) $\dfrac{x}{1+x^2}$.

7. The **Fundamental Theorem of Calculus** says, in part, that *if f is continuous on (a,b) then $F(x) = \int_a^x f(t)\,dt$ is a primitive for $f(x)$ on (a,b).*

Prove that $\ln x$, defined earlier as the inverse of e^x, satisfies

$$\ln x = \int_1^x \frac{dt}{t}$$

for all $x > 0$.

Hint: Find two primitives for $f : (0,\infty) \to \mathbb{R}, x \longmapsto 1/x$ and note that primitives are unique up to a constant.

Additional Questions

8. Integrate $f(x) = x^2 - 6x + 10$ over $[2, 5]$ using the arithmetic partition of $[2, 5]$ into $3n$ equal length subintervals.

   Note how we look at $\mathcal{P}_{3n}$ and not $\mathcal{P}_n$, ask yourself why.

9. Let $f : [0, 1] \to \mathbb{R}$ be given by $f(0) = 0$ and, for $x \in (0, 1]$,

   $$f(x) = \frac{1}{n} \text{ where } n \text{ is the largest integer satisfying } n \leq \frac{1}{x}.$$

   Draw the graph of f. Show that f is monotonic on $[0, 1]$.

   Deduce that f is Riemann integrable on $[0, 1]$.

   Find

   $$\int_0^1 f.$$

   Hint. First calculate the integral over $[1/N, 1]$ for any $N \geq 1$. Then use this in evaluating the upper and lower integrals of f over $[0, 1]$.